Fact: (1). $V(0) = \mathbb{A}^n$, $V(1) = \phi$,
(2). $\bigcap_\alpha V(I_\alpha) = V(\bigcup_\alpha I_\alpha)$
(3). $V(I) \cup V(J) = V(IJ)$
(4). $V(x_1-a_1, x_2-a_2, \cdots, x_n-a_n) = \{(a_1, a_2, \cdots, a_n)\}$

$\Rightarrow$ To top. on $\mathbb{A}^n$

Zariski top.

a Topology on a set X:

$$\mathcal{T} \subseteq \{ \text{subsets of } X \}$$

- $\phi, X \in \mathcal{T}$
- $\{U_i\}_{i \in I} \subseteq \mathcal{T} \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$
- $\{U_1, \cdots, U_n\} \subseteq \mathcal{T} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

The subsets of X in $\mathcal{T}$ are called open subsets.
The complement of an open subset is called closed subset in X.

Fact: $\{ \mathbb{A}^n \setminus V(I) \mid I \triangleleft k[x_1, \cdots, x_n] \} \subseteq \{ \text{subsets of } \mathbb{A}^n \}$ forms a Topology of $\mathbb{A}^n$, called Zariski top.

Fact: A subsets $Z \subseteq \mathbb{A}^n$ is closed under Zariski top iff Z is algebraic

§1.3. The Ideal of a Set of points.

$\forall\, X \subseteq \mathbb{A}^n(k)$

$$I(X) := \{ F \in k[x_1, \cdots, x_n] \mid F(a) = 0 \ \forall a \in X \}$$

ideal of X

Example:
1) $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \subseteq \mathbb{A}^1(\mathbb{C})$
2) $X = \{ (x, y) \mid y = \sin x \} \subseteq \mathbb{A}^2(\mathbb{R})$
3) $I(\phi) = k[x_1, \cdots, x_n]$ & $I(\mathbb{A}^n(k)) = (0)$. (Assume $\#k = \infty$)
4) $I(\{(a_1, \cdots, a_n)\}) = (x_1 - a_1, \cdots, x_n - a_n) \quad \forall\, a_1, \cdots, a_n \in k$.

Fact:
(1) $X \subset Y \Rightarrow I(X) \supset I(Y)$
(2) $I(V(S)) \supset S$ & $V(I(X)) \supset X$
(3) $V(I(V(S))) = V(S)$. $\quad I(V(I(X))) = I(X)$ $\Rightarrow$
(4) $I(X)$ is a radical ideal. i.e. $I(X) = \sqrt{I(X)}$

$$\left(\bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I} \right)$$

$I = \sqrt{I} \Leftrightarrow I$ is int of some prim idls

全体代数集 $\qquad$ 全体?

$$\operatorname{Im} V \overset{1:1}{\longleftrightarrow} \operatorname{Im} I$$

$\Downarrow$

推论?

几何 $\qquad\qquad\qquad\qquad$ 代数

$$\begin{tikzcd}
\{ \text{Subsets in } \mathbb{A}^n(k) \} \arrow[drr, "I"] & & \{ \text{ideals} \} \arrow[dll, "V"'] \\
\cup & & \cup \\
\{ \text{alg. subsets in } \mathbb{A}^n(k) \} \arrow[rr, shift right, "I"'] & & \{ \text{Radical ideals} \} \arrow[ll, shift right, "V"']
\end{tikzcd}$$

$x^2 + y^2 \in \mathbb{R}[x, y]$

Fact: V, W = alg. Then

$$I(V) = I(W) \Rightarrow V = W$$

Pf: Assume $V = V(S_1)$, $W = V(S_2)$.

$$V = V(S_1) = V \circ \underline{I \circ V(S_1)} = V \circ \underline{I \circ V(S_2)} = V(S_2) = W.$$

Fact: $P \notin V \Rightarrow I(V) \neq I(V \cup \{P\})$!

e.g. $V = \{P_1, \cdots, P_n\}$ $\exists f$ s.t. $f(P_1) = \cdots = f(P_n) = 0 \neq f(P)$

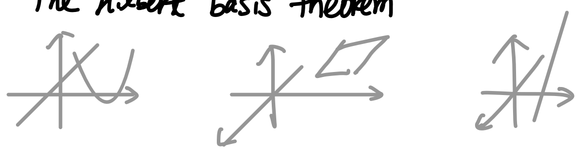
Rmk: $k = \bar{k} \Rightarrow$ bijection in above diagram.

一般代数集有哪些, 如何研究.

1) 从代数表达入手 (对应理想)

2) 从几何图形入手 (对应的拓扑)

§1.4 The Hilbert basis theorem



多个方程. 有限个?

Thm 1. Every algebraic set is a finite intersection of hypersurfaces.

Noeth ring = a ring with every ideal being f.g.

Thm (Hilbert basis thm): R = noeth. $\Rightarrow$ $R[x_1, \cdots, x_n]$ = noeth.

pf: WMA: $n=1$. (i.e. R = noeth $\Rightarrow$ $R[x]$ = noeth.)

$\forall m \geq 0$, $J_m := \{$leading coeff. of poly. of deg $\leq m\} \triangleleft R$

$J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$

R = noeth $\Rightarrow$ $J_\cdot$ stable (i.e. $\exists N$ s.t. $J_N = J_{N+1} = \cdots$)

Choose $F_{i1}, F_{i2}, \cdots F_{ik_i}$ s.t. the leading coefs generate J_i. Then

Claim: I is generated by $\left\{ F_{ij} \,\middle|\, \begin{matrix} 0 \leq i \leq N \\ 1 \leq j \leq k_i \end{matrix} \right\}$

... To be added.

Cor: $k[x_1, \cdots, x_n]$ = noeth. & $k[x_1, \cdots, x_n]/I$ = noeth.

every algebraic set is defined by a finite number of polynomial.

(12) pf of thm1: ...